ON THE ANALYTIC DESIGN OF CONTROLS IN SYSTEMS WITH RANDOM CHARACTERISTICS

(OB ANALITICHESKOM KONSTRUIROVANII REGULIATOROV V SISTEMAKH SO SLUCHAINYMI SVOISTVAMI)

PMM Vol. 26, No. 2, 1962, pp. 259-266

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(Received November 18, 1961)

Problems of constructing an optimal controller in a stochastic linear system for a condition of minimum least-square error, are investigated. The construction of an optimal Liapunov function [1] by the method of small parameters [2] is described. The paper continues the researches of [3, 4].

1. Let the transient response of a control system be described by the equation

$$\frac{dx}{dt} = A(\eta)x + c(\eta)\xi \tag{1.1}$$

Here x and c are n-vectors, $\eta(t)$ is a random variable, $A(\eta)$ is a matrix of the form $\| a_{ij} \|_1^n$; the scalar $\xi(n, \eta)$ represents the control action (the control).

As in [4] we describe the Markov process $\eta(t)$ by means of the functions $q(\alpha)$ and $q(\alpha, \beta)$ in the following way [5]

$$P[\eta(t+\Delta t) = \alpha / \eta(t) = \alpha] = 1 - q(\alpha)\Delta t + o(\Delta t)$$
$$P[\eta(t+\Delta t] \neq \alpha,\ \eta(t+\Delta t) \leqslant \beta / \eta(t) = \alpha] = q(\alpha, \beta)\Delta t + o(\Delta t)$$

where P is the conditional probability.

We shall call the control $\xi(x, \eta)$ optimal with respect to system (1.1) if it ensures a minimum mean value of the least-square error integral

$$J = M\int_0^\infty \Big(\sum_n x_i^2 + \xi^2\Big)dt$$

The way of constructing ξ is based on the Liapunov function method with the use of the dynamic programming method [6] for the stochastic system (1.1). We use below the concepts and notations introduced in [4]. The Liapunov function can be chosen in the form

$$v(x, \eta) = \sum_{i,j}^{n} b_{ij}(\eta) x_i x_j$$

the coefficients of which are obtained for $\eta \in [\eta_1, \eta_2]$ by solving a system of quadratic integral equations. The optimal control is found from the condition

$$\xi = -\frac{1}{2} \sum_{i=j}^{n} c_i(\eta) \frac{\partial v}{\partial x_i} \tag{1.2}$$

In general, the solution of a system of quadratic integral equations is difficult. In this paper we study two cases where the problem can be solved effectively by the method of small parameters. For this the functions $v(x, \eta)$ and $\xi(x, \eta)$ can be expressed as series in powers of a parameter μ:

$$v(x, \eta, \mu) = \sum_{k=0}^{\infty} \mu^k v_k, \qquad \xi(x, \eta, \mu) = \sum_{k=0}^{\infty} \mu^k \xi_k \tag{1.3}$$

(where the necessity of solving the above-mentioned system of quadratic integral equations is eliminated).

The problem is reduced to the successive computation of the coefficients v_k and ξ_k in (1.3) from linear systems. The convergence of the series so obtained for $v(x, \eta)$ and $\xi(x, \eta)$ is proved.

Case 1.1. The probability of the transition $\eta = \alpha \to \eta = \beta$ is small, i.e.

$$q(\alpha) = \mu r(\alpha), \qquad q(\alpha, \beta) = \mu r(\alpha, \beta) \tag{1.4}$$

Here μ is a small parameter.

Case 1.2. Equation (1.1) can be written in the form

$$dx/dt = Ax + \mu R(\eta) x + c\xi \tag{1.5}$$

Here $\mu R(\eta)x$ is a group of terms depending on $\eta(t)$, μ is a small parameter, $R(\eta)$ is a matrix of the form $\| r_{ij} \|_1^n$.

2. We study Case 1.1. Let $\mu = 0$. Then the function $v(x, \eta)$ will be equal to the zero coefficient v_0 of series (1.3). Further, by virtue of (1.4) we have $q(\alpha) = q(\alpha, \beta) = 0$, and, consequently, the problem is reduced to the determination of the Liapunov function v_0 and the optimal control ξ_0 for every fixed value $\eta = \gamma$ for a determinate system of the

form

$$dx / dt = A(\gamma) x + c(\gamma) \xi_0 \tag{2.1}$$

where ξ_0 is the zero coefficient in series (1.3).

Such a problem is investigated in [3] where a method is indicated of determining the coefficients $b_{ij}^{(0)}(\gamma)$ of the positive definite form which is the Liapunov function for (2.1). For completeness we mention here the system of equations determining $b_{ij}^{(0)}$

$$-\Big[\sum_{i=1}^{n} c_i(\gamma) b_{ki}^{(0)}(\gamma)\Big]\Big[\sum_{i=1}^{n} c_i(\gamma) b_{si}^{(0)}(\gamma)\Big] +$$

$$+ \sum_{i=1}^{n} [b_{ki}^{(0)}(\gamma) a_{is}(\gamma) + b_{si}^{(0)}(\gamma) a_{ik}(\gamma)] = \begin{cases} 0 & (k \neq s) \\ -1 & (k = s) \end{cases}$$

and, in addition

$$\xi_0 = -\frac{1}{2} \sum_{i=1}^{n} c_i(\gamma) \frac{\partial v_0}{\partial x_i} = -\sum_{i=1}^{n} c_i(\gamma) \Big[\sum_{j=1}^{n} b_{ij}^{(0)}(\gamma) x_j\Big]$$

The sufficient condition for the existence of functions v_0 and ξ_0 is the linear independence of the vectors $c, Ac, \ldots, A^{n-1}c$ [7]. Let us assume that this condition is satisfied uniformly for η. Then v_0 and ξ_0 can be determined uniquely for any $\eta \in [\eta_1, \eta_2]$.

We prove that after computing v_0 and ξ_0 we can successively determine the coefficients of series (1.3) for every fixed $\eta = \gamma$ by solving a system of linear algebraic equations.

The dynamic programming equations for the investigation of the problem with respect to (1.3) and (1.4) for $\eta = \gamma$, have the form [4]

$$\left(\frac{dM\{v\}}{dt}\right)_{\eta=\gamma} = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} \Big[\sum_{j=1}^{n} a_{ij} x_j + c_i \xi\Big] +$$

$$+ \mu \int_{\eta_1}^{\eta_2} [v(x, \lambda) - v(x, \gamma)] \, d_\lambda r(\gamma, \lambda) = -\sum_{n} x_i^2 - \xi^2 \tag{2.2}$$

$$\xi = -\frac{1}{2} \sum_{i=1}^{n} c_i \Big(\sum_{k=0}^{\infty} \mu^k \frac{\partial v_k}{\partial x_i}\Big) \tag{2.3}$$

The meaning of the derivative $(dM\{v\}/dt)_{\eta=\gamma}$, taken with respect to Equation (1.1), and the method of its computation, are given in [4,9]. The integral in Equation (2.2) is a Stieltjes integral.

From (2.3) we obtain the expression for the series coefficients

$$\xi_k = -\frac{1}{2}\sum_{i=1}^{n} c_i(\gamma)\frac{\partial v_k(x,\gamma)}{\partial x_i} \tag{2.4}$$

We substitute series (1.3) into Equation (2.2) and equate the coefficients of like powers of the parameter μ. As a result of the transformation, taking (2.4) into account, we have

$$\sum_{i=1}^{n}\frac{\partial v_k(x,\gamma)}{\partial x_i}\left[c_i(\gamma)\xi_0 + \sum_{j=1}^{n} a_{ij}(\gamma)x_j\right] =$$
$$= \sum_{s=1}^{k-1}\xi_s\xi_{k-s} + \int_{\eta_2}^{\eta_1}[v_{k-1}(x,\lambda) - v_{k-1}(x,\gamma)]\,d_\lambda r(\gamma,\lambda) \tag{2.5}$$

The left-hand side of (2.5) is the derivative $dv_k(x,\gamma)/dt$ with respect to system (2.1) for $\eta = \gamma$. We designate it by $(dv_k/dt)_{\eta=\gamma}$. Then

$$\left(\frac{dv_k}{dt}\right)_{\eta=\gamma} = \sum_{s=1}^{k-1}\xi_s\xi_{k-s} + \int_{\eta_2}^{\eta_1}[v_{k-1}(x,\lambda) - v_{k-1}(x,\gamma)]\,d_\lambda r(\gamma,\lambda) \tag{2.6}$$

Equations (2.4) and (2.6) allow us to find the kth coefficient $v_k(x,\eta)$ for every $\eta = \gamma$ if we know the $k-1$ preceding coefficients. Since the right-hand side of (2.6) is a certain quadratic form, and system (2.1) is asymptotically stable, then by [8, p.61] there exists a unique solution of (2.6) for $\eta = \gamma$ which is the quadratic form $v_k(x, \eta)$:

$$v_k(x,\eta) = \sum_{i,j}^{n} b_{ij}^{(k)}(\eta)x_ix_j \tag{2.7}$$

Substituting (2.7) into (2.6) and equating the coefficients for the product $x_m x_l$, we obtain a system of linear algebraic equations for determining $b_{ml}^{(k)}$

$$\sum_{i=1}^{n} b_{im}^{(k)}\left[a_{il} + c_i\sum_{j=1}^{n} c_j b_{jl}^{(0)}\right] + \sum_{i=1}^{n} b_{il}^{(k)}\left[a_{im} + c_i\sum_{j=1}^{n} c_j b_{jm}^{(0)}\right] =$$
$$= 2\sum_{s=1}^{k-1}\left[\sum_{i,j}^{n} c_ic_jb_{jm}^{(s)}b_{il}^{(k-s)}\right] + 2\int_{\eta_2}^{\eta_1}[b_{ml}^{(k-1)}(\lambda) - b_{ml}^{(k-1)}(\gamma)]\,d_\lambda r(\gamma,\lambda) \tag{2.8}$$

We investigate Case 1.2. For $\mu = 0$ system (1.5) can be written in the form

$$\frac{dx}{dt} = Ax + c\xi_0 \tag{2.9}$$

where A is a constant matrix, c is a constant vector, ξ_0 is the zero coefficient in series (1.3).

Therefore, in this case the problem reduces to the determination of the zero coefficients of series (1.3) to obtain the Liapunov function v_0 and the optimal control ξ_0 for the determinate system (2.9).

The order of computing v_0 and ξ_0 is indicated above [3].

Substituting (2.5) here, the computation reduces to the equation

$$\sum_{i=1}^{n} \frac{\partial v_k}{\partial x_i}\Big[\sum_{j=1}^{n} a_{ij}x_j + c_i\xi_0\Big] + \int_{\eta_1}^{\eta_2} [v_k(x, \lambda) - v_k(x, \gamma)]\, d_\lambda q(\gamma, \lambda) = \\ = \sum_{s=1}^{k-1} \xi_s\xi_{k-s} - \sum_{i=1}^{n} \frac{\partial v_{k-1}}{\partial x_i}\Big(\sum_{j=1}^{n} r_{ij}x_j\Big) \qquad (2.10)$$

The left-hand side of (2.10) is the derivative $(dM\{v_k\}/dt)_{\eta=\gamma}$, taken with respect to Equation (2.9) [4,9]:

$$\left(\frac{dM\{v_k\}}{dt}\right)_{\eta-\gamma} = \sum_{s=1}^{k-1} \xi_s\xi_{k-s} - \sum_{i=1}^{n}\Big(\sum_{j=1}^{n} r_{ij}x_j\Big)\frac{\partial v_{k-1}}{\partial x_i} \qquad (2.11)$$

Taking (2.4) into account from (2.11) we obtain a system of linear integral equations for determining the coefficients

$$\sum_{i=1}^{n} b_{im}^{(k)}\Big[a_{il} + c_i\sum_{j=1}^{n} c_j b_{jl}^{(0)}\Big] + \sum_{i=1}^{n} b_{il}^{(k)}\Big[a_{im} + c_i\sum_{j=1}^{n} c_j b_{jm}^{(0)}\Big] + \\ + 2\int_{\eta_1}^{\eta_2} [b_{ml}^{(k)}(\lambda) - b_{ml}^{(k)}(\gamma)]\, d_\lambda q(\gamma, \lambda) = \\ = 2\sum_{s=1}^{k-1}\sum_{i,j}^{n} [c_i c_j b_{im}^{(s)} b_{jl}^{(k-s)}] - 2\sum_{i=1}^{n} [b_{im}^{(k-1)} r_{il} + b_{il}^{(k-1)} r_{im}] \qquad (2.12)$$

If function $q(\alpha, \beta)$ assumes the density

$$p(\alpha, \beta) = \sum_{i=1}^{m} f_i(\alpha)\varphi_i(\beta)$$

then system (2.12) will consist of equations with a degenerate kernel, for which an effective method of solution is known [10].

3. We will prove that under the assumed conditions series (1.3) will converge.

(1) From the study of Case 1.1 we find an estimate for the coefficient v_k of the first series in (1.3) in terms of the estimates of the coefficients $v_0, \ldots, v_{k-1}$ of the same series.

From Equations (2.4) and (2.6) it follows that the right-hand side of (2.6) is a quadratic form

$$\left(\frac{dv_k}{dt}\right)_{\eta=\gamma} = \sum_{i,j}^{n} \beta_{ij}^{(k)} x_i x_j \tag{3.1}$$

where the coefficients $\beta_{ml}^{(k)}$ equal the right-hand sides of the corresponding equations of system (2.8):

$$\beta_{ml}^{(k)} = 2\sum_{s=1}^{k-1}\left[\sum_{i,j}^{n} c_i c_j b_{jm}^{(s)} b_{il}^{(k-s)}\right] + 2\int_{\eta_2}^{\eta_1} [b_{ml}^{(k-1)}(\lambda) - b_{ml}^{(k-1)}(\gamma)]\, d_\lambda r(\gamma, \lambda) \tag{3.2}$$

From (3.1) follows

$$v_k(T) - v_k(0) = \int_0^T \left(\sum_{i,j}^{n} \beta_{ij}^{(k)} x_i x_j\right) dt$$

Since by hypothesis system (1.1) is asymptotically stable, then as $T \to \infty$ the form $v_k(T) \to 0$

$$v_k(0) = -\int_0^\infty \left(\sum_{i,j}^{n} \beta_{ij}^{(k)} x_i x_j\right) dt \tag{3.3}$$

Let us denote by $\| F_{ij}(t) \|_1^n$ the matrix of the normal fundamental system of solutions of the original Equation (1.1) for $\eta = \gamma$.

We determine the solutions of the system of Equations (1.1) by the Cauchy formula for homogeneous systems [11]:

$$x_i(t) = \sum_{j=1}^{n} F_{ij}(t)\, x_{j0} \tag{3.4}$$

For the asymptotic stability of the system for $\eta = \gamma$ the equality [8]

$$|F_{ij}(t)| < Be^{-\alpha(t-t_0)} \tag{3.5}$$

should be valid, in which the numbers B and α can, in general, be chosen independently of the initial instant t_0 and of the fixed value of $\eta = \gamma$.

Let us assume $t_0 = 0$.

From (3.3) to (3.5) the inequality

$$|v_k(x, \gamma)| < \int_0^\infty \sum_{i,j}^n \left\{ |\beta_{ij}^{(k)}| B^2 e^{-2\alpha t} \sum_{p,q}^n |x_p x_q| \right\} dt \tag{3.6}$$

is correct.

From the theory of quadratic forms we have

$$\sum_{p,q}^n |x_p x_q| = \left(\sum_{m=1}^n |x_m| \right)^2 \leqslant n \sum_{m=1}^n x_m^2 \tag{3.7}$$

Therefore

$$|v_k(x, \gamma)| < \frac{n}{2\alpha} B^2 \left(\sum_{i,j}^n |\beta_{ij}^{(k)}| \right) \left(\sum_{m=1}^n x_m^2 \right) \tag{3.8}$$

We assume that for every value $s = 0, 1, \ldots, k-1$ the coefficients $b_{ij}^{(s)}$ are computed. We introduce the notation

$$\max |b_{ij}^{(s)}| = L_s, \qquad \max |c_i c_j| = N \tag{3.9}$$

For the integral on the right-hand side of (3.2) we can make the estimate

$$\int_{\eta_2}^{\eta_1} | [\beta_{ml}^{(k-1)}(\lambda) - b_{ml}^{(k-1)}(\gamma)] | \, d_\lambda r(\gamma, \lambda) \leqslant 2L_{k-1} r(\gamma)$$

The estimates for all the coefficients $\beta_{ij}^{(k)}$ have the form

$$|\beta_{ij}^{(k)}| < 2Nn^2 \sum_{s=1}^{k-1} L_s L_{k-s} + 4L_{k-1} r(\gamma) \tag{3.10}$$

We choose for the form $v(x, \gamma)$ a number $c > 0$ such that the inequality

$$\max |v_k(x, \gamma)| \geqslant c L_k \sum_n x_m^2 \tag{3.11}$$

is satisfied.

It is obvious that the coefficient c can be chosen independently of the number k.

Using (3.6) to (3.11), for L_k we have an estimate in terms of the known numbers $L_s (s = 0, 1, \ldots, k-1)$

$$cL_k < \frac{B^2Nn^5}{\alpha}\sum_{s=1}^{k-1} L_sL_{k-s} + 4n^2L_{k-1}r_{\max} \qquad (r_{\max} = \max r(\eta)) \tag{3.12}$$

For any k and η the numbers c, B, α, N can, in general, be chosen independently of k and η. We write (3.12) in the form

$$L_k < A\sum_{s=1}^{k-1} L_sL_{k-s} + AL_{k-1} \tag{3.13}$$

Here

$$A = \max A_i \quad (i = 1,\ 2), \qquad A_1 = \frac{B^2Nn^5}{\alpha c} > 0, \qquad A_2 = \frac{4n^2r_{\max}}{c} > 0$$

The remainder of the proof consists of applying the method of dominating series [12]. We investigate the quadratic equation

$$\rho^2 + (a + \mu)\rho + b = 0 \tag{3.14}$$

where a, b, μ are certain numbers.

If the number μ is sufficiently small, the roots of (3.14) can always be written in the form of a convergent series in μ

$$\rho^{(1,\,2)} = -\frac{a+\mu}{2} \pm \sqrt{\left(\frac{a+\mu}{2}\right)^2 - b} = \sum_{k=0}^{\infty} \mu^k\rho_k \tag{3.15}$$

We will prove that for definite values of a and b in (3.15), one of the roots $\rho^{(1)}$, $\rho^{(2)}$ can be represented by a series dominating series (1.3).

We substitute (3.15) into (3.14) and equate the coefficients of μ^k ($k = 0,\ 1,\ \ldots$) to zero. As a result we obtain an expression for ρ_k in terms of $\rho_0, \ldots, \rho_{k-1}$

$$\rho_k = -\frac{1}{2\rho_0 + a}\left(\sum_{s=1}^{k-1} \rho_s\rho_{k-s} + \rho_{k-1}\right) \tag{3.16}$$

The value of ρ_0 is found from the equation

$$\rho_0^2 + a\rho_0 + b = 0 \tag{3.17}$$

From (3.16) and (3.13) it follows that the constructed convergent series (3.15) will dominate series (1.3) if we take

$$\rho_0 = L_0, \qquad -\frac{1}{2\rho_0 + a} = A$$

Hence the values of the coefficients a and b in Equation (3.14) will be

$$a = -\frac{1 + 2L_0 A}{A} < 0, \qquad b = \frac{1 + L_0 A}{A} L_0 > 0$$

By knowing a and b, from (3.17) we find that the dominant for (1.3) is that root of (3.14) the value of which is determined by the zero coefficient

$$\rho_0 = L_0 = -\frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - b}$$

The convergence of (1.3) is proved.

(2) Case 1.2 can be investigated in analogous order.

Equation (2.11) is written in the form

$$\left(\frac{dM\{v_k\}}{dt}\right)_{\eta=\gamma} = \sum_{i,j}^{n} \beta_{ij}^{(k)} x_i x_j \tag{3.18}$$

Here the coefficients of the form, $\beta_{ml}^{(k)}$, equal the right-hand sides of the corresponding equations of system (2.12).

We will prove [4] that the equality

$$v(x, \gamma) = -\int_0^\infty M\left\{\sum_{i,j}^{n} \beta_{ij}^{(k)} x_i x_j / x, \gamma\right\} dt \tag{3.19}$$

follows from (3.18).

Here the symbol $M\{S/x, \gamma\}$ is the mean value of variable S for initial conditions x, γ.

We investigate the derivative $dM\{v_k(x(t), \eta(t))/x_0, \gamma, t_0\}/dt$, taken with respect to Equation (2.9). Averaging with respect to x, η we get the equality

$$\frac{dM\{v_k(x, \eta)/x_0, \gamma, t_0\}}{dt} = M\left\{\frac{dM\{v_k\}}{dt} \Big/ x_0, \gamma, t_0\right\} = M\left\{\sum_{i,j}^{n} \beta_{ij}^{(k)} x_i x_j / x_0, \gamma, t_0\right\}$$

By integration we have

$$M\{v_k(x(T), \eta(T))/x_0, \gamma, t_0\} - v(x_0, \gamma) = \int_0^T M\left\{\sum_{i,j}^{n} \beta_{ij}^{(k)} x_i x_j / x_0, \gamma\right\} dt$$

Since the system is asymptotically stable, $M\{v_k\} \to 0$ as $T \to \infty$, which proves (3.19).

Dropping the transformations similar to those we had used earlier, we write an estimate for the coefficients v_k of series (1.3)

$$|v_k(x, \gamma)| < \frac{n}{2\alpha} B^2 \Big(\sum_n x_m^2\Big) M \Big\{\sum_{i,j}^{n} |\beta_{ij}^{(\omega)}| / \eta = \gamma\Big\}$$

If we designate $\delta = \max|r_{ij}(\eta)|$ for $\eta_1 \leqslant \eta \leqslant \eta_2$, then, in the light of notation (3.9), we get

$$cL_k < \frac{B^2 n^5 N}{\alpha} \sum_{s=1}^{k-1} L_s L_{k-s} + 4n^3 \delta L_{k-1}$$

The remainder of the proof repeats the reasoning set forth above.

Remark 3.1. The construction of a dominating series allows us not only to prove the convergence of series (1.3) but also to determine the radius of convergence in a known fashion.

We formulate the results obtained as a theorem.

Theorem 3.1. If for system (1.1) in the interval $\eta_1 \leqslant \eta \leqslant \eta_2$, the coefficients $A(\eta)$ and $c(\eta)$ are continuous and the following conditions are satisfied: 1) the system of vectors $c(\eta)$, $A(\eta)c(\eta)$, ..., $A^{n-1}(\eta)c(\eta)$ is linearly independent, 2) either the probability of the transition $\eta = \alpha \to \eta = \beta$ is small or the right-hand side of (1.1) can be written in the form (1.5), then, the Liapunov function $v(x, \eta)$ and the optimal control $\xi(x, \eta)$ can be represented in the form of series (1.3). The coefficients of these series can be found by solving linear systems of algebraic or integral equations.

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Translated by N.H.C.